LIFTING RETRACTED DIAGRAMS WITH RESPECT TO PROJECTABLE FUNCTORS

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ABSTRACT. We prove a general categorical theorem that enables us to state that under certain conditions, the range of a functor is large. As an application, we prove various results of which the following is a prototype: *If every diagram, indexed by a lattice, of finite* Boolean $\langle \vee, 0 \rangle$-*semilattices with* $\langle \vee, 0 \rangle$-*embeddings, can be lifted with respect to the* $\mathrm{Con_c}$ *functor on lattices, then so can every diagram, indexed by a lattice, of finite* distributive $\langle \vee, 0 \rangle$-*semilattices with* $\langle \vee, 0 \rangle$-*embeddings.* If the premise of this statement held, this would solve in turn the (still open) problem whether every distributive algebraic lattice is isomorphic to the congruence lattice of a lattice. We also outline potential applications of the method to other functors, such as the $R \mapsto V(R)$ functor on von Neumann regular rings.

1. INTRODUCTION

We are dealing with the following general kind of problem. We are given a functor $\mathbf{F}$ from a category $\mathcal{A}$ to a category $\mathcal{B}$, we wish to investigate the range of $\mathbf{F}$, that is, the class of all objects of $\mathcal{B}$ isomorphic to some $\mathbf{F}(A)$, where A is an object of $\mathcal{A}$. The goal of the present paper is to provide a general categorical framework for proving results saying that if certain diagrams of $\mathcal{B}$ can be lifted with respect to $\mathbf{F}$, then the range of $\mathbf{F}$ is large.

Let us be more precise. We assume that $\mathcal{B}$ has a subcategory $\mathcal{B}'$, in which many diagrams can be lifted with respect to $\mathbf{F}$. Furthermore, we assume that every member of $\mathcal{B}$ is a *retract* of some member of $\mathcal{B}'$, and this in a functorial way; we say that $\mathcal{B}$ is a *functorial retract* of $\mathcal{B}'$ (see Definition 2.2). Then, under certain conditions, many more diagrams in $\mathcal{B}$ (not only in $\mathcal{B}'$) can be lifted with respect to $\mathbf{F}$. This is the main result of the paper, Theorem 6.3. It is stated in the language of category theory.

We do not challenge the view that Theorem 6.3 is a technical, at first sight unappealing result, whose formulation lies apparently far away from most readers' usual mathematical practice. However, the main motivation for establishing it lies in an easily formulated problem, namely R. P. Dilworth's Congruence Lattice Problem

CLP, open since 1945, that asks whether any distributive algebraic lattice is iso-
morphic to the congruence lattice of a lattice, see G. Grätzer and E. T. Schmidt [10]
or J. Tůma and F. Wehrung [22] for surveys. Theorem 6.3 makes it possible to re-
duce a certain diagrammatic strengthening of CLP to diagrams of *finite Boolean*
semilattices and $\langle \vee, 0 \rangle$-*embeddings*, see Proposition 7.8. This reduction requires the
result, established by the author in [31], that states that *every finite distributive*
$\langle \vee, 0 \rangle$-*semilattice is a* $\langle \vee, 0 \rangle$-*retract, in a functorial way, of some finite Boolean*
$\langle \vee, 0 \rangle$-*semilattice.* Moreover, the categorical formulation of Theorem 6.3 makes it
possible to attack other representation problems from a new perspective: for exam-
ple, functorial lifting, with respect to the compact congruence semilattice functor
$\mathrm{Con_c}$, of finite $\langle \vee, 0 \rangle$-semilattices by algebras of a given similarity type (see The-
orem 7.4 and Proposition 7.6), or representation problem of countable refinement
monoids by the nonstable K-theory of von Neumann regular rings (see Proposi-
tion 8.2). Although no definite answer of the corresponding long-standing open
problems is reached here, we hope that the ideas of the present paper will provide
a new start towards more final conclusions.

We conclude the paper by mentioning further functors for which our results
might be of some potential use, together with a few open problems.

While the present paper deals with how to *use* functorial retractions in order to
prove that certain functors have large range, the paper [31] deals with the *existence*
of functorial retractions.

As for [31], the techniques introduced in the present paper were developed to
solve problems closer to universal algebra than to category theory. Nevertheless, the
paper is, up to Section 6, purely category-theoretical. For these reasons, the author
chose to write it in probably more detail than a category theorist would wish, with
the hope to make it reasonably intelligible to members of both communities.

2. Basic concepts

We shall identify every natural number n with the set $\{0, 1, \ldots, n-1\}$. Similarly,
$\omega = \{0, 1, 2, \ldots\}$, the set of all natural numbers.

We shall mostly use the notation and terminology from S. Mac Lane [13] and
from the author's previous paper [31]. A morphism in a given category is an *epic*, if
it is right cancellable; we shall often denote epics by double arrows, as in $f \colon A \twoheadrightarrow B$.

The following definitions from [31] will be crucial.

Definition 2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be subcategories of a category $\mathcal{C}$. We denote by
$\mathrm{Retr}(\mathcal{A}, \mathcal{B})$ the category whose objects and morphisms are the following:

— *Objects*: all quadruples $\langle A, B, \varepsilon, \mu \rangle$, where $A \in \mathrm{Ob}\,\mathcal{A}$, $B \in \mathrm{Ob}\,\mathcal{B}$, $\varepsilon \colon A \to B$,
$\mu \colon B \to A$, and $\mu \circ \varepsilon = \mathrm{id}_A$.
— *Morphisms*: a morphism from $\langle A, B, \varepsilon, \mu \rangle$ to $\langle A', B', \varepsilon', \mu' \rangle$ is a pair $\langle f, g \rangle$,
where $f \colon A \to A'$ in $\mathcal{A}$, $g \colon B \to B'$ in $\mathcal{B}$, $g \circ \varepsilon = \varepsilon' \circ f$, and $\mu' \circ g =
f \circ \mu$. Composition of morphisms is defined by the rule $\langle f', g' \rangle \circ \langle f, g \rangle =
\langle f' \circ f, g' \circ g \rangle$.

The *projection functor* from $\mathrm{Retr}(\mathcal{A}, \mathcal{B})$ to $\mathcal{A}$ is the functor from $\mathrm{Retr}(\mathcal{A}, \mathcal{B})$ to $\mathcal{A}$
that sends any object $\langle A, B, \varepsilon, \mu \rangle$ to A and any morphism $\langle f, g \rangle$ to f.

Definition 2.2. We say that $\mathcal{A}$ is a *functorial retract* of $\mathcal{B}$, if the projection functor
from $\mathrm{Retr}(\mathcal{A}, \mathcal{B})$ to $\mathcal{A}$ has a right inverse. We shall call such an inverse a *functorial
retraction* of $\mathcal{A}$ to $\mathcal{B}$.

Hence a functorial retraction may be viewed as a triple $\langle \Phi, \varepsilon, \mu \rangle$ that satisfies the following conditions:

— Φ is a functor from $\mathcal{A}$ to $\mathcal{B}$.
— For every morphism $f \colon X \to Y$ in $\mathcal{A}$, we have $\varepsilon_X \colon X \to \Phi(X)$, $\mu_X \colon \Phi(X) \to X$, $\mu_X \circ \varepsilon_X = \mathrm{id}_X$, $\Phi(f) \circ \varepsilon_X = \varepsilon_Y \circ f$, and $\mu_Y \circ \Phi(f) = f \circ \mu_X$.

We shall often deal with colimits with respect to upward directed posets, in short *directed colimits* (often called *direct limits* in other mathematical areas). We say that a category $\mathcal{A}$ has *directed ω-colimits*, if every diagram of $\mathcal{A}$ of the form

$$A_0 \xrightarrow{\ f_0\ } A_1 \xrightarrow{\ f_1\ } A_2 \xrightarrow{\ f_2\ } \cdots \ \cdots$$

(indexed by the poset ω) has a colimit in $\mathcal{A}$. For a category $\mathcal{B}$, we say that a functor $\mathbf{F} \colon \mathcal{A} \to \mathcal{B}$ *preserves directed ω-colimits*, if whenever $A = \varinjlim_{n < \omega} A_n$, with transition morphisms $f_n \colon A_n \to A_{n+1}$ and limiting morphisms $g_n \colon A_n \to A$ (for $n < \omega$), the relation $\mathbf{F}(A) = \varinjlim_{n < \omega} \mathbf{F}(A_n)$ holds, with transition morphisms $\mathbf{F}(f_n)$ and limiting morphisms $\mathbf{F}(g_n)$ (for $n < \omega$).

The following easy result is in some sense the central idea that makes it possible in this paper to express objects as colimits of "simpler" objects.

Lemma 2.3. *Let $\mathcal{B}$ be a category and let $\langle A, B, \varepsilon, \mu \rangle$ be an object of $\mathrm{Retr}(\mathcal{B}, \mathcal{B})$. Put $\rho = \varepsilon \circ \mu$. Then A is the colimit of the sequence*

$$B \xrightarrow{\ \rho\ } B \xrightarrow{\ \rho\ } B \xrightarrow{\ \rho\ } \cdots \ \cdots$$

with constant limiting morphism $\mu \colon B \to A$.

Proof. Clearly, $\rho \circ \rho = \rho$ and $\mu \circ \rho = \mu$. Now let C be an object of $\mathcal{B}$ and let $\langle \varphi_n \mid n < \omega \rangle$ be a sequence of morphisms, $\varphi_n \colon B \to C$, such that $\varphi_n = \varphi_{n+1} \circ \rho$ for all $n < \omega$. Since ρ is idempotent, $\varphi_n = \varphi_0$ for all $n < \omega$. The morphism $\varphi_0 \circ \varepsilon$ is the unique morphism $\psi \colon A \to C$ such that $\psi \circ \mu = \varphi_0$. $\qquad\square$

We shall use the following notation for *products*. For objects A, B, and C in a given category, the notation $C = A \Pi B$ (wrt. a, b) means that C is the product of A and B with projections $a \colon C \to A$ and $b \colon C \to B$. For an object D and morphisms $f \colon D \to A$ and $g \colon D \to B$, we shall denote by $f \times g$ the unique morphism $h \colon D \to C$ such that $f = a \circ h$ and $g = b \circ h$. In case $D = A' \Pi B'$ (wrt. a', b'), $f \colon A' \to A$, and $g \colon B' \to B$, we shall put $f \Pi g = (f \circ a') \times (g \circ b')$.

If $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{I}$ are categories, a *lifting* of a functor $\mathbf{D} \colon \mathcal{I} \to \mathcal{B}$ with respect to a functor $\mathbf{F} \colon \mathcal{A} \to \mathcal{B}$ is a functor $\mathbf{E} \colon \mathcal{I} \to \mathcal{A}$ such that the composition $\mathbf{F}\mathbf{E}$ is equivalent to $\mathbf{D}$.

Our commutative monoids will be denoted additively. Let M be a commutative monoid. We say that M is a *refinement monoid* (see H. Dobbertin [3] or F. Wehrung [24, 25]), if $a_0 + a_1 = b_0 + b_1$ in M implies that there are elements $c_{i,j} \in M$ (for i, $j < 2$) such that $a_i = c_{i,0} + c_{i,1}$ and $b_i = c_{0,i} + c_{1,i}$, for all $i < 2$. We say that M is *conical*, if $x + y = 0$ implies that $x = y = 0$, for all $x, y \in M$. It is well-known that a $\langle \vee, 0 \rangle$-semilattice is a refinement monoid iff it is distributive, see G. Grätzer [9, Section II.5]. Every commutative monoid is endowed with its *algebraic quasi-ordering*, defined by $x \leq y \Leftrightarrow (\exists z)(x + z = y)$. A monoid homomorphism $f \colon M \to N$ is an *embedding*, if it is one-to-one and $f(x) \leq f(y)$ iff $x \leq y$, for all x, $y \in M$. An *order-unit* of M is an element $u \in M$ such that for all $x \in M$, there exists a positive integer n such that $x \leq nu$.

A subset I of a commutative monoid M is an *o-ideal*, if $0 \in I$ and $x + y \in I$ iff $x \in I$ and $y \in I$, for all x, $y \in M$. For x, $y \in M$, let $x \equiv_I y$ hold, if there are u, $v \in I$ such that $x + u = y + v$. Then $\equiv_I$ is a monoid congruence on M; we put $M/I = M/\equiv_I$. A monoid homomorphism $f \colon M \to N$ between commutative monoids M and N is *ideal-induced*, if f is *surjective* and the o-ideal $I = f^{-1}\{0\}$ satisfies the condition that $f(x) = f(y)$ iff $x \equiv_I y$, for all x, $y \in M$. Equivalently, the homomorphism f is isomorphic, in the category of homomorphisms with domain M, to the canonical projection $M \twoheadrightarrow M/I$.

3. Projecting morphisms with respect to projectable functors

Definition 3.1. In a category $\mathcal{C}$, a morphism $f \colon A \to B$ is a *projection*, if there exists a decomposition of the form $A = B \amalg B'$ (wrt. f, f') in $\mathcal{C}$.

Definition 3.2. Let $\mathbf{F}$ be a functor from a category $\mathcal{A}$ to a category $\mathcal{B}$, let $A \in \operatorname{Ob}\mathcal{A}$, $B \in \operatorname{Ob}\mathcal{B}$, and $\varphi \colon \mathbf{F}(A) \to B$. A *projectability witness* for $\langle \varphi, A, B \rangle$ (with respect to $\mathbf{F}$) is a pair $\langle a, \varepsilon \rangle$ satisfying the following conditions:

   (i) $a \colon A \twoheadrightarrow \overline{A}$ is an epic in $\mathcal{A}$.
   (ii) $\varepsilon \colon \mathbf{F}(\overline{A}) \to B$ is an isomorphism in $\mathcal{B}$.
   (iii) $\varphi = \varepsilon \circ \mathbf{F}(a)$.
   (iv) For every $f \colon A \to X$ in $\mathcal{A}$ and every $\eta \colon \mathbf{F}(\overline{A}) \to \mathbf{F}(X)$ such that $\mathbf{F}(f) = \eta \circ \mathbf{F}(a)$, there exists $g \colon \overline{A} \to X$ in $\mathcal{A}$ such that $f = g \circ a$ and $\eta = \mathbf{F}(g)$.

We say that $\mathbf{F}$ is *projectable*, if every triple $\langle \varphi, A, B \rangle$, where $\varphi \colon \mathbf{F}(A) \to B$ is a *projection*, has a projectability witness.

We observe that the morphism g in (iii) above is necessarily *unique* (because a is an epic). Furthermore, the projectability witness $\langle a, \varepsilon \rangle$ is unique up to isomorphism.

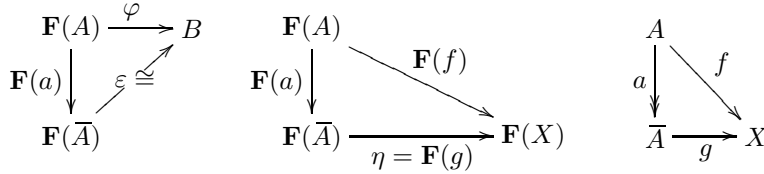Definition 3.2 is illustrated on Figure 3.1.

Figure 3.1. Projectability of the functor $\mathbf{F}$.

4. From $\mathbb{D} \colon \mathcal{I} \to \operatorname{Retr}(\mathcal{B}, \mathcal{B})$ to the unfolding $\widetilde{\mathbb{D}} \colon \mathcal{I} \times \omega \to \mathcal{B}$

In this section we shall fix categories $\mathcal{I}$ and $\mathcal{B}$, together with a functor $\mathbb{D} \colon \mathcal{I} \to \operatorname{Retr}(\mathcal{B}, \mathcal{B})$ (see Definition 2.1). We also assume that $\mathcal{B}$ has nonempty finite products. (Equivalently, $B_0 \amalg B_1$ exists, for all objects B_0 and B_1 of $\mathcal{B}$.) We shall describe a construction that creates a functor $\widetilde{\mathbb{D}} \colon \mathcal{I} \times \omega \to \mathcal{B}$, that we shall call the *unfolding* of $\mathbb{D}$.

The functor $\mathbb{D}$ is given by functors $\mathbf{D}$, $\widehat{\mathbf{D}} \colon \mathcal{I} \to \mathcal{B}$, together with correspondences $X \mapsto \varepsilon_X$ and $X \mapsto \mu_X$, for $X \in \operatorname{Ob}\mathcal{I}$, such that the following conditions hold for all $f \colon X \to Y$ in $\mathcal{I}$ (see Figure 4.1):

   (i) $\mu_X \circ \varepsilon_X = \operatorname{id}_{\mathbf{D}(X)}$;

(ii) $\widehat{\mathbf{D}}(f) \circ \varepsilon_X = \varepsilon_Y \circ \mathbf{D}(f)$ and $\mu_Y \circ \widehat{\mathbf{D}}(f) = \mathbf{D}(f) \circ \mu_X$.

$$
\begin{array}{ccc}
\widehat{\mathbf{D}}(X) & \xrightarrow{\;\widehat{\mathbf{D}}(f)\;} & \widehat{\mathbf{D}}(Y) \\
\varepsilon_X \big\uparrow\big\downarrow \mu_X & & \varepsilon_Y \big\uparrow\big\downarrow \mu_Y \\
\mathbf{D}(X) & \xrightarrow[\;\mathbf{D}(f)\;]{} & \mathbf{D}(Y)
\end{array}
$$

FIGURE 4.1. Description of the functor $\mathbb{D}$.

In particular, $\rho_X = \varepsilon_X \circ \mu_X$ is an idempotent endomorphism of $\widehat{\mathbf{D}}(X)$. We shall need the following lemma.

Lemma 4.1. *For any $f \colon X \to Y$ in $\mathfrak{I}$, the equality $\widehat{\mathbf{D}}(f) \circ \rho_X = \rho_Y \circ \widehat{\mathbf{D}}(f)$ holds.*

Proof. We compute $\widehat{\mathbf{D}}(f) \circ \varepsilon_X \circ \mu_X = \varepsilon_Y \circ \mathbf{D}(f) \circ \mu_X = \varepsilon_Y \circ \mu_Y \circ \widehat{\mathbf{D}}(f)$. $\qquad \square$

For $X \in \mathrm{Ob}\,\mathfrak{I}$, we define $\widehat{\mathbf{D}}^n(X)$ by induction on n, by putting

$$\widehat{\mathbf{D}}^1(X) = \widehat{\mathbf{D}}(X) \tag{4.1}$$

$$\widehat{\mathbf{D}}^{n+1}(X) = \widehat{\mathbf{D}}(X) \amalg \widehat{\mathbf{D}}^n(X) \quad (\text{wrt. } \alpha_{n+1}^X, \pi_{n+1}^X) \tag{4.2}$$

We also put $\alpha_1^X = \mathrm{id}_{\widehat{\mathbf{D}}(X)}$, while π_1^X is undefined. We also put

$$\sigma_n^X = (\rho_X \circ \alpha_n^X) \times \mathrm{id}_{\widehat{\mathbf{D}}^n(X)},$$

with respect to the decomposition (4.2) (see the left half of Figure 4.2).

If $f \colon X \to Y$ in $\mathfrak{I}$, we define $\widehat{\mathbf{D}}^n(f)$ by induction on n, by putting $\widehat{\mathbf{D}}^1(f) = \widehat{\mathbf{D}}(f)$ and $\widehat{\mathbf{D}}^{n+1}(f) = \widehat{\mathbf{D}}(f)\amalg\widehat{\mathbf{D}}^n(f)$, with respect to $\alpha_{n+1}^X, \pi_{n+1}^X$. Hence $\widehat{\mathbf{D}}^n(f) \colon \widehat{\mathbf{D}}^n(X) \to \widehat{\mathbf{D}}^n(Y)$ in $\mathcal{B}$ (see the right half of Figure 4.2).
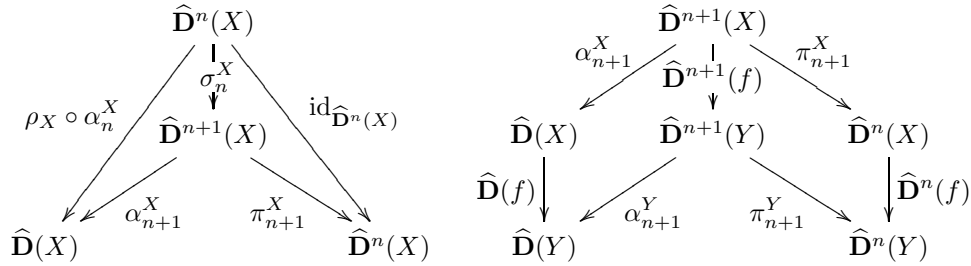

FIGURE 4.2. The morphisms σ_n^X and $\widehat{\mathbf{D}}^n(f)$.

The following lemma is a trivial consequence of the relations on the left half of Figure 4.2.

Lemma 4.2. *For any $X \in \mathrm{Ob}\,\mathfrak{I}$ and $n \in \omega \setminus \{0\}$, the equality $\alpha_{n+1}^X \circ \sigma_n^X = \rho_X \circ \alpha_n^X$ holds.*

Lemma 4.3. *For any $f \colon X \to Y$ in $\mathcal{I}$ and $n \in \omega \setminus \{0\}$, the equality $\alpha_n^Y \circ \widehat{\mathbf{D}}^n(f) = \widehat{\mathbf{D}}(f) \circ \alpha_n^X$ holds.*

Proof. For $n = 1$ it is trivial. At stage $n + 1$, with $n > 0$, the conclusion follows from the relations on the right half of Figure 4.2. $\qquad \square$

Lemma 4.4. *For $f \colon X \to Y$ in $\mathcal{I}$ and $n \in \omega \setminus \{0\}$, the equality $\widehat{\mathbf{D}}^{n+1}(f) \circ \sigma_n^X = \sigma_n^Y \circ \widehat{\mathbf{D}}^n(f)$ holds.*

Proof. By the definition of the statement (4.2), it suffices to prove that the compositions of the desired equality on the left with both α_{n+1}^Y and π_{n+1}^Y hold. Composition with α_{n+1}^Y: we calculate, using Figure 4.2,

$$
\begin{aligned}
\alpha_{n+1}^Y \circ \widehat{\mathbf{D}}^{n+1}(f) \circ \sigma_n^X &= \widehat{\mathbf{D}}(f) \circ \alpha_{n+1}^X \circ \sigma_n^X \\
&= \widehat{\mathbf{D}}(f) \circ \rho_X \circ \alpha_n^X \\
&= \rho_Y \circ \widehat{\mathbf{D}}(f) \circ \alpha_n^X && \text{(by Lemma 4.1)} \\
&= \rho_Y \circ \alpha_n^Y \circ \widehat{\mathbf{D}}^n(f) && \text{(by Lemma 4.3)} \\
&= \alpha_{n+1}^Y \circ \sigma_n^Y \circ \widehat{\mathbf{D}}^n(f).
\end{aligned}
$$

Composition with π_{n+1}^Y: we calculate, using Figure 4.2,

$$
\begin{aligned}
\pi_{n+1}^Y \circ \widehat{\mathbf{D}}^{n+1}(f) \circ \sigma_n^X &= \widehat{\mathbf{D}}^n(f) \circ \pi_{n+1}^X \circ \sigma_n^X \\
&= \widehat{\mathbf{D}}^n(f) \\
&= \pi_{n+1}^Y \circ \sigma_n^Y \circ \widehat{\mathbf{D}}^n(f).
\end{aligned}
$$

The conclusion follows. $\qquad \square$

It follows from Lemma 4.4 that the diagram of Figure 4.3 is commutative, for any $f \colon X \to Y$ in $\mathcal{I}$.

FIGURE 4.3. A subdiagram of the functor $\widetilde{\mathbb{D}}$.

We construct the functor $\widetilde{\mathbb{D}} \colon \mathcal{I} \times \omega \to \mathcal{B}$ so that all the diagrams represented by Figure 4.3 are subdiagrams of $\widetilde{\mathbb{D}}$. More precisely:

— $\widetilde{\mathbb{D}}(X, n) = \widehat{\mathbf{D}}^{n+1}(X)$, for all $\langle X, n \rangle \in \mathrm{Ob}\,\mathcal{I} \times \omega$.
— If $f \colon X \to Y$ in $\mathcal{I}$ and $m \leq n < \omega$, let

$$
\widetilde{\mathbb{D}}(f, (m \to n)) = \sigma_n^Y \circ \cdots \circ \sigma_{m+1}^Y \circ \widehat{\mathbf{D}}^{m+1}(f).
$$

It follows from Lemma 4.4 (that is, the commutativity of the diagram represented on Figure 4.3) that we also have

$$\widetilde{\mathbb{D}}(f, (m \to n)) = \sigma_n^Y \circ \cdots \circ \sigma_{i+1}^Y \circ \widehat{\mathbf{D}}^{i+1}(f) \circ \sigma_i^X \circ \cdots \circ \sigma_{m+1}^X,$$

for all $i \in \{m, m+1, \ldots, n\}$.

5. The functor $\mathbf{Q}^n$ and the equivalence ζ

In this section we shall fix again categories $\mathcal{I}$ and $\mathcal{B}$, the latter having nonempty finite products, together with a functor $\mathbb{D} \colon \mathcal{I} \to \mathrm{Retr}(\mathcal{B}, \mathcal{B})$. As in Section 4, let $\mathbb{D}$ be given by $\left\langle \mathbf{D}, \widehat{\mathbf{D}}, \varepsilon, \mu \right\rangle$.

We assume that the unfolding $\widetilde{\mathbb{D}}$ (see Section 4) has a lifting $\mathbb{E}$ with respect to $\mathbf{F}$. For $f \colon X \to Y$ in $\mathcal{I}$ and $n \in \omega \setminus \{0\}$, we put

$$\mathbf{E}^n(X) = \mathbb{E}(X, n-1),$$
$$s_n^X = \mathbb{E}(\mathrm{id}_X, (n-1 \to n)),$$
$$\mathbf{E}^n(f) = \mathbb{E}(f, (n-1 \to n-1)),$$

and we let the isomorphisms $\eta_n^X \colon \mathbf{FE}^n(X) \to \widehat{\mathbf{D}}^n(X)$ witness the equivalence between $\mathbf{FE}$ and $\widetilde{\mathbb{D}}$. The various relations between these morphisms can be read on the commutative diagrams represented on Figure 5.1.

FIGURE 5.1. The lifting $\mathbb{E}$ and the equivalence η.

Let $X \in \mathrm{Ob}\,\mathcal{I}$ and let $n \in \omega \setminus \{0\}$. Since the functor $\mathbf{F}$ is projectable and α_n^X is either an isomorphism or a projection (thus so is $\alpha_n^X \circ \eta_n^X$), the triple $\left\langle \alpha_n^X \circ \eta_n^X, \mathbf{E}^n(X), \widehat{\mathbf{D}}(X) \right\rangle$ has a projectability witness, say, $\left\langle a_n^X, \zeta_n^X \right\rangle$, with an epic $a_n^X \colon \mathbf{E}^n(X) \twoheadrightarrow \mathbf{Q}^n(X)$ and an isomorphism $\zeta_n^X \colon \mathbf{FQ}^n(X) \to \widehat{\mathbf{D}}(X)$ (see Figure 5.2).

FIGURE 5.2. The projectability witness $\left\langle a_n^X, \zeta_n^X \right\rangle$ for $\left\langle \alpha_n^X \circ \eta_n^X, \mathbf{E}^n(X), \widehat{\mathbf{D}}(X) \right\rangle$.

Lemma 5.1. *For all $X \in \mathrm{Ob}\,\mathfrak{I}$ and all $n \in \omega \setminus \{0\}$, there exists a unique morphism $\overline{s}_n^X \colon \mathbf{Q}^n(X) \to \mathbf{Q}^{n+1}(X)$ such that $\overline{s}_n^X \circ a_n^X = a_{n+1}^X \circ s_n^X$ and $\zeta_{n+1}^X \circ \mathbf{F}(\overline{s}_n^X) = \rho_X \circ \zeta_n^X$.*

The meaning of Lemma 5.1 is illustrated on Figure 5.3.

$$
\begin{array}{ccc}
\mathbf{E}^n(X) \xrightarrow{\; s_n^X \;} \mathbf{E}^{n+1}(X) & \qquad & \mathbf{FQ}^n(X) \xrightarrow{\; \mathbf{F}(\overline{s}_n^X) \;} \mathbf{FQ}^{n+1}(X) \\
{\scriptstyle a_n^X}\big\downarrow \qquad \big\downarrow{\scriptstyle a_{n+1}^X} & & {\scriptstyle \zeta_n^X}\big\downarrow \qquad \big\downarrow{\scriptstyle \zeta_{n+1}^X} \\
\mathbf{Q}^n(X) \xrightarrow[\; \overline{s}_n^X \;]{} \mathbf{Q}^{n+1}(X) & & \widehat{\mathbf{D}}(X) \xrightarrow[\; \rho_X \;]{} \widehat{\mathbf{D}}(X)
\end{array}
$$

FIGURE 5.3. The morphism $\overline{s}_n^X$.

Proof. Put $\psi = (\zeta_{n+1}^X)^{-1} \circ \rho_X \circ \zeta_n^X$, so $\psi \colon \mathbf{FQ}^n(X) \to \mathbf{FQ}^{n+1}(X)$. We compute:

$$
\begin{aligned}
\psi \circ \mathbf{F}(a_n^X) &= (\zeta_{n+1}^X)^{-1} \circ \rho_X \circ \zeta_n^X \circ \mathbf{F}(a_n^X) \\
&= (\zeta_{n+1}^X)^{-1} \circ \rho_X \circ \alpha_n^X \circ \eta_n^X && \text{(see Figure 5.2)} \\
&= (\zeta_{n+1}^X)^{-1} \circ \alpha_{n+1}^X \circ \sigma_n^X \circ \eta_n^X && \text{(see Lemma 4.2)} \\
&= \mathbf{F}(a_{n+1}^X) \circ (\eta_{n+1}^X)^{-1} \circ \sigma_n^X \circ \eta_n^X && \text{(see Figure 5.2)} \\
&= \mathbf{F}(a_{n+1}^X) \circ \mathbf{F}(s_n^X) && \text{(see Figure 5.1)} \\
&= \mathbf{F}(a_{n+1}^X \circ s_n^X).
\end{aligned}
$$

Hence, since $\left\langle a_n^X, \zeta_n^X \right\rangle$ is a projectability witness for $\left\langle \alpha_n^X \circ \eta_n^X, \mathbf{E}^n(X), \widehat{\mathbf{D}}(X) \right\rangle$, there exists a unique morphism $\overline{s}_n^X \colon \mathbf{Q}^n(X) \to \mathbf{Q}^{n+1}(X)$ such that $\overline{s}_n^X \circ a_n^X = a_{n+1}^X \circ s_n^X$ and $\mathbf{F}(\overline{s}_n^X) = \psi$. The latter condition is equivalent to $\zeta_{n+1}^X \circ \mathbf{F}(\overline{s}_n^X) = \rho_X \circ \zeta_n^X$. $\qquad \square$

Lemma 5.2. *Let $f \colon X \to Y$ in $\mathfrak{I}$ and let $n \in \omega \setminus \{0\}$. There exists a unique morphism $\mathbf{Q}^n(f) \colon \mathbf{Q}^n(X) \to \mathbf{Q}^n(Y)$ such that $\mathbf{Q}^n(f) \circ a_n^X = a_n^Y \circ \mathbf{E}^n(f)$ and $\zeta_n^Y \circ \mathbf{FQ}^n(f) = \widehat{\mathbf{D}}(f) \circ \zeta_n^X$.*

The meaning of Lemma 5.2 is illustrated on Figure 5.4.

$$
\begin{array}{ccc}
\mathbf{E}^n(X) \xrightarrow{\; \mathbf{E}^n(f) \;} \mathbf{E}^n(Y) & \qquad & \mathbf{FQ}^n(X) \xrightarrow{\; \mathbf{FQ}^n(f) \;} \mathbf{FQ}^n(Y) \\
{\scriptstyle a_n^X}\big\downarrow \qquad \big\downarrow{\scriptstyle a_n^Y} & & {\scriptstyle \zeta_n^X}\big\downarrow \qquad \big\downarrow{\scriptstyle \zeta_n^Y} \\
\mathbf{Q}^n(X) \xrightarrow[\; \mathbf{Q}^n(f) \;]{} \mathbf{Q}^n(Y) & & \widehat{\mathbf{D}}(X) \xrightarrow[\; \widehat{\mathbf{D}}(f) \;]{} \widehat{\mathbf{D}}(Y)
\end{array}
$$

FIGURE 5.4. The morphism $\mathbf{Q}^n(f)$.

Proof. Put $\psi = (\zeta_n^Y)^{-1} \circ \widehat{\mathbf{D}}(f) \circ \zeta_n^X$, so $\psi \colon \mathbf{F}\mathbf{Q}^n(X) \to \mathbf{F}\mathbf{Q}^n(Y)$. We compute:

$$
\begin{aligned}
\psi \circ \mathbf{F}(a_n^X) &= (\zeta_n^Y)^{-1} \circ \widehat{\mathbf{D}}(f) \circ \zeta_n^X \circ \mathbf{F}(a_n^X) \\
&= (\zeta_n^Y)^{-1} \circ \widehat{\mathbf{D}}(f) \circ \alpha_n^X \circ \eta_n^X && \text{(see Figure 5.2)} \\
&= (\zeta_n^Y)^{-1} \circ \alpha_n^Y \circ \widehat{\mathbf{D}}^n(f) \circ \eta_n^X && \text{(see Lemma 4.3)} \\
&= \mathbf{F}(a_n^Y) \circ (\eta_n^Y)^{-1} \circ \widehat{\mathbf{D}}^n(f) \circ \eta_n^X && \text{(see Figure 5.2)} \\
&= \mathbf{F}(a_n^Y) \circ \mathbf{F}\mathbf{E}^n(f) && \text{(see Figure 5.1)} \\
&= \mathbf{F}\big(a_n^Y \circ \mathbf{E}^n(f)\big).
\end{aligned}
$$

Hence, since $\big\langle a_n^X, \zeta_n^X \big\rangle$ is a projectability witness for $\big\langle \alpha_n^X \circ \eta_n^X, \mathbf{E}^n(X), \widehat{\mathbf{D}}(X) \big\rangle$, there exists a unique morphism $\mathbf{Q}^n(f) \colon \mathbf{Q}^n(X) \to \mathbf{Q}^n(Y)$ such that $\mathbf{Q}^n(f) \circ a_n^X = a_n^Y \circ \mathbf{E}^n(f)$ and $\mathbf{F}\mathbf{Q}^n(f) = \psi$. The latter condition is equivalent to $\zeta_n^Y \circ \mathbf{F}\mathbf{Q}^n(f) = \widehat{\mathbf{D}}(f) \circ \zeta_n^X$. $\qquad\square$

Lemma 5.3. *For any $n \in \omega \setminus \{0\}$, the correspondences $\mathbf{Q}^n$ (on objects and morphisms) define a functor from $\mathcal{I}$ to $\mathcal{A}$. Furthermore, for any $f \colon X \to Y$ in $\mathcal{I}$, the equality $\mathbf{Q}^{n+1} \circ \overline{s}_n^X = \overline{s}_n^Y \circ \mathbf{Q}^n(f)$ holds (see Figure 5.5).*

Lemma 5.3 means that the correspondences $\langle X, n \rangle \mapsto \mathbf{Q}^{n+1}(X)$ and $\langle f, n \rangle \mapsto \mathbf{Q}^{n+1}(f)$ define a functor from $\mathcal{I} \times \omega$ to $\mathcal{A}$.

$$
\begin{array}{ccc}
\mathbf{Q}^n(Y) & \xrightarrow{\ \overline{s}_n^Y\ } & \mathbf{Q}^{n+1}(Y) \\[4pt]
{\scriptstyle \mathbf{Q}^n(f)}\big\uparrow & & \big\uparrow{\scriptstyle \mathbf{Q}^{n+1}(f)} \\[4pt]
\mathbf{Q}^n(X) & \xrightarrow[\ \overline{s}_n^X\ ]{} & \mathbf{Q}^{n+1}(X)
\end{array}
$$

FIGURE 5.5. Extending $\mathbf{Q}$ to a functor from $\mathcal{I} \times \omega$ to $\mathcal{A}$.

Proof. We verify that $\mathbf{Q}^n(g \circ f) = \mathbf{Q}^n(g) \circ \mathbf{Q}^n(f)$, for all $f \colon X \to Y$ and $g \colon Y \to Z$ in $\mathcal{I}$. We compute:

$$
\begin{aligned}
\mathbf{Q}^n(g) \circ \mathbf{Q}^n(f) \circ a_n^X &= \mathbf{Q}^n(g) \circ a_n^Y \circ \mathbf{E}^n(f) && \text{(see Lemma 5.2)} \\
&= a_n^Z \circ \mathbf{E}^n(g) \circ \mathbf{E}^n(f) && \text{(see Lemma 5.2)} \\
&= a_n^Z \circ \mathbf{E}^n(g \circ f) && \text{(because $\mathbf{E}^n$ is a functor)} \\
&= \mathbf{Q}^n(g \circ f) \circ a_n^X.
\end{aligned}
$$

Since a_n^X is an epic, we obtain that $\mathbf{Q}^n(g) \circ \mathbf{Q}^n(f) = \mathbf{Q}^n(g \circ f)$. It is clear that $\mathbf{Q}^n$ preserves identities, hence it is a functor. We now prove the additional equality:

$$
\begin{aligned}
\mathbf{Q}^{n+1}(f) \circ \overline{s}_n^X \circ a_n^X &= \mathbf{Q}^{n+1}(f) \circ a_{n+1}^X \circ s_n^X && \text{(see Lemma 5.1)} \\
&= a_{n+1}^Y \circ \mathbf{E}^{n+1}(f) \circ s_n^X && \text{(see Lemma 5.2)} \\
&= a_{n+1}^Y \circ s_n^Y \circ \mathbf{E}^n(f) && \text{(see Figure 5.1)} \\
&= \overline{s}_n^Y \circ a_n^Y \circ \mathbf{E}^n(f) && \text{(see Lemma 5.1))} \\
&= \overline{s}_n^Y \circ \mathbf{Q}^n(f) \circ a_n^X. && \text{(see Lemma 5.2)}
\end{aligned}
$$

Therefore, since a_n^X is an epic, $\mathbf{Q}^{n+1}(f) \circ \overline{s}_n^X = \overline{s}_n^Y \circ \mathbf{Q}^n(f)$.   □

6. CONSTRUCTING A LIFTING OF $\mathbf{D}$

In the present section we shall make, until the statement of Theorem 6.3, the same assumptions as in Section 5. Furthermore, we assume that $\mathcal{A}$ has directed ω-colimits and that $\mathbf{F}$ preserves directed ω-colimits (see Section 2). We keep the notations of Section 5, in particular for $\mathbf{Q}^n$ and ζ_n^X.

For any $X \in \mathrm{Ob}\,\mathfrak{I}$, let $\mathbf{R}(X)$ denote the colimit of the sequence

$$\mathbf{Q}^1(X) \xrightarrow{\overline{s}_1^X} \mathbf{Q}^2(X) \xrightarrow{\overline{s}_2^X} \mathbf{Q}^3(X) \xrightarrow{\overline{s}_3^X} \cdots \cdots, \tag{6.1}$$

with limiting morphisms $t_n^X \colon \mathbf{Q}^n(X) \to \mathbf{R}(X)$. Hence $t_n^X = t_{n+1}^X \circ \overline{s}_n^X$, for all $n \in \omega \setminus \{0\}$ (see Figure 6.1).

Furthermore, it follows from Lemma 2.3 that $\mathbf{D}(X)$ is the colimit of the sequence

$$\widehat{\mathbf{D}}(X) \xrightarrow{\rho_X} \widehat{\mathbf{D}}(X) \xrightarrow{\rho_X} \widehat{\mathbf{D}}(X) \xrightarrow{\rho_X} \cdots \cdots$$

with constant limiting morphism $\mu_X \colon \widehat{\mathbf{D}}(X) \to \mathbf{D}(X)$. Now we apply $\mathbf{F}$ to the diagram (6.1), together with its colimit and limiting morphisms. Since $\mathbf{F}$ preserves directed ω-colimits and by Lemma 5.1 (see Figure 5.3), there exists a unique isomorphism $\delta^X \colon \mathbf{FR}(X) \to \mathbf{D}(X)$ such that $\delta^X \circ \mathbf{F}(t_n^X) = \mu_X \circ \zeta_n^X$, for all $n \in \omega \setminus \{0\}$ (see Figure 6.1).

FIGURE 6.1. Defining $\mathbf{R}(X)$, t_n^X, and δ^X.

We shall prove that $\mathbf{R}$ is a lifting of $\mathbf{D}$, with category equivalence δ.

First, let $f \colon X \to Y$ in $\mathfrak{I}$. By the universal property of the colimit and by Lemma 5.3 (see Figure 5.5), there exists a unique morphism $\mathbf{R}(f) \colon \mathbf{R}(X) \to \mathbf{R}(Y)$ such that $\mathbf{R}(f) \circ t_n^X = t_n^Y \circ \mathbf{Q}^n(f)$ for all $n \in \omega \setminus \{0\}$ (see Figure 6.2).

FIGURE 6.2. Defining $\mathbf{R}(f)$.

Lemma 6.1. *The correspondence* $X \mapsto \mathbf{R}(X)$, $f \mapsto \mathbf{R}(f)$ *defines a functor from* $\mathfrak{I}$ *to* $\mathcal{A}$.

Proof. As $\mathbf{R}$ obviously preserves identities, it is sufficient to verify that $\mathbf{R}(g \circ f) = \mathbf{R}(g) \circ \mathbf{R}(f)$ whenever $f \colon X \to Y$ and $g \colon Y \to Z$ in $\mathfrak{I}$. For all $n \in \omega \setminus \{0\}$, we compute, by using the definitions of $\mathbf{R}(f)$ and $\mathbf{R}(g)$:

$$
\begin{aligned}
\mathbf{R}(g) \circ \mathbf{R}(f) \circ t_n^X &= \mathbf{R}(g) \circ t_n^Y \circ \mathbf{Q}^n(f) \\
&= t_n^Z \circ \mathbf{Q}^n(g) \circ \mathbf{Q}^n(f) \\
&= t_n^Z \circ \mathbf{Q}^n(g \circ f) \qquad \text{(see Lemma 5.3)},
\end{aligned}
$$

whence, by the definition of $\mathbf{R}(g \circ f)$, we obtain the desired equality. $\qquad\square$

Lemma 6.2. *The correspondence $X \mapsto \delta^X$ defines a category equivalence from the composition $\mathbf{FR}$ to $\mathbf{D}$.*

In particular, $\mathbf{R}$ is a lifting of $\mathbf{D}$ with respect to $\mathbf{F}$.

Proof. We need to prove that the equality $\delta^Y \circ \mathbf{FR}(f) = \mathbf{D}(f) \circ \delta^X$ holds, for any $f \colon X \to Y$ in $\mathfrak{I}$. Put $\varphi = \delta^Y \circ \mathbf{FR}(f)$ and $\psi = \mathbf{D}(f) \circ \delta^X$. For all $n \in \omega \setminus \{0\}$, we compute:

$$
\begin{aligned}
\varphi \circ \mathbf{F}(t_n^X) = \delta^Y \circ \mathbf{F}(t_n^Y) \circ \mathbf{FQ}^n(f) &\qquad \text{(see Figure 6.2)} \\
= \mu_Y \circ \zeta_n^Y \circ \mathbf{FQ}^n(f) &\qquad \text{(see Figure 6.1)} \\
= \mu_Y \circ \widehat{\mathbf{D}}(f) \circ \zeta_n^X &\qquad \text{(see Lemma 5.2)} \\
= \mathbf{D}(f) \circ \mu_X \circ \zeta_n^X &\qquad \text{(see Figure 4.1)} \\
= \psi \circ \mathbf{F}(t_n^X) &\qquad \text{(see Figure 6.1)}.
\end{aligned}
$$

As $\mathbf{FR}(X) = \varinjlim_n \mathbf{FQ}^n(X)$ with limiting morphisms $\mathbf{F}(t_n^X) \colon \mathbf{FQ}^n(X) \to \mathbf{FR}(X)$, it follows from the uniqueness statement in the definition of the colimit that $\varphi = \psi$. $\qquad\square$

Hence we have completed the proof of our main technical result.

Theorem 6.3. *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathfrak{I}$ be categories, together with functors $\mathbf{F} \colon \mathcal{A} \to \mathcal{B}$ and $\mathbb{D} \colon \mathfrak{I} \to \mathrm{Retr}(\mathcal{B}, \mathcal{B})$. We denote by Π the projection functor from $\mathrm{Retr}(\mathcal{B}, \mathcal{B})$ to $\mathcal{B}$, and we assume the following:*

 (i) *The category $\mathcal{A}$ has directed ω-colimits.*
 (ii) *The category $\mathcal{B}$ has nonempty finite products.*
 (iii) *The functor $\mathbf{F}$ preserves directed ω-colimits.*
 (iv) *The functor $\mathbf{F}$ is projectable.*

If the unfolding $\widetilde{\mathbb{D}}$ of $\mathbb{D}$ (see Section 4) has a lifting with respect to $\mathbf{F}$, then so does the composition $\Pi\mathbb{D}$.

Observe that with the notation of Section 4, $\Pi\mathbb{D} = \mathbf{D}$.

We will always use Theorem 6.3 in the following way. For functors $\mathbf{F} \colon \mathcal{A} \to \mathcal{B}$ and $\mathbf{D} \colon \mathfrak{I} \to \mathcal{B}$, we first extend $\mathbf{D}$ to a functor $\mathbb{D} \colon \mathfrak{I} \to \mathrm{Retr}(\mathcal{B}, \mathcal{B})$, in the sense that $\Pi\mathbb{D} = \mathbf{D}$. This can sometimes be done by using results in [31] (as in the proof of Theorem 7.4(ii) in the present paper). Then, for the associated functor $\widehat{\mathbf{D}} \colon \mathfrak{I} \to \mathcal{B}$, the vertices of the unfolding $\widetilde{\mathbb{D}}$ are finite powers of the form $\widehat{\mathbf{D}}^n(X)$, where $X \in \mathrm{Ob}\,\mathfrak{I}$ and $n \in \omega \setminus \{0\}$. Hence Theorem 6.3 reduces lifting of diagrams with vertices in $\mathcal{B}$ to lifting of diagrams whose vertices are finite powers of objects in the range of $\widehat{\mathbf{D}}$. In the context of Theorem 7.4(ii), the former are finite distributive $\langle \vee, 0 \rangle$-semilattices, while the latter are finite Boolean $\langle \vee, 0 \rangle$-semilattices.

7. Lifting diagrams of $\langle \vee, 0 \rangle$-semilattices with respect to the compact congruence functor on a variety

For unexplained definitions of universal algebra we refer the reader to G. Grätzer [8] or S. Burris and H. P. Sankappanavar [2].

Let $\mathcal{V}$ be a variety of algebras in a similarity type Σ. So $\mathcal{V}$, endowed with its homomorphisms, becomes a category. This category has many nice properties, in particular, it has all small limits and small colimits. For a member A of $\mathcal{V}$, let $\mathrm{Con}_c\, A$ denote the semilattice of compact (i.e., finitely generated) congruences of A, endowed with join. Then $\mathrm{Con}_c\, A$ is a $\langle \vee, 0 \rangle$-semilattice. Furthermore, if $f \colon A \to B$ is a homomorphism in $\mathcal{V}$, one can define a $\langle \vee, 0 \rangle$-homomorphism $\mathrm{Con}_c\, f \colon \mathrm{Con}_c\, A \to \mathrm{Con}_c\, B$ by

$$(\mathrm{Con}_c\, f)(\boldsymbol{a}) = \text{congruence of } B \text{ generated by } \{\langle f(x), f(y) \rangle \mid \langle x, y \rangle \in \boldsymbol{a}\},$$

for any $\boldsymbol{a} \in \mathrm{Con}_c\, A$. These rules determine a functor, still denoted by Con_c, from $\mathcal{V}$ to the category $\mathcal{S}$ of $\langle \vee, 0 \rangle$-semilattices and $\langle \vee, 0 \rangle$-homomorphisms. This functor preserves directed colimits.

The key point that relates this context to our general categorical framework is the following.

Lemma 7.1. *Let A be an algebra in $\mathcal{V}$, let S be a $\langle \vee, 0 \rangle$-semilattice, and let $\varphi \colon \mathrm{Con}_c\, A \twoheadrightarrow S$ be an ideal-induced $\langle \vee, 0 \rangle$-homomorphism (see Section 2). Then the triple $\langle \varphi, A, S \rangle$ has a projectability witness with respect to the functor Con_c. In particular, the functor Con_c is projectable.*

Proof. Put $I = \varphi^{-1}\{0\}$ and denote by $\boldsymbol{a}$ the congruence of A defined by

$$\langle x, y \rangle \in \boldsymbol{a} \iff \varphi \Theta_A(x, y) = 0, \text{ for all } x, y \in A,$$

where $\Theta_A(x, y)$ denotes the congruence of A generated by the pair $\langle x, y \rangle$. Put $\overline{A} = A/\boldsymbol{a}$ and denote by $p \colon A \twoheadrightarrow \overline{A}$ the canonical projection. Since p is surjective, it is an epic.

Claim. *For all $\boldsymbol{u} \in \mathrm{Con}_c\, A$, $\varphi(\boldsymbol{u}) = 0$ iff $\boldsymbol{u} \subseteq \boldsymbol{a}$.*

Proof of Claim. Suppose first that $\varphi(\boldsymbol{u}) = 0$, and let $x, y \in A$ such that $x \equiv y \pmod{\boldsymbol{u}}$. This can be written $\Theta_A(x, y) \subseteq \boldsymbol{u}$, thus, applying φ and the assumption, $\varphi \Theta_A(x, y) = 0$, that is, $x \equiv y \pmod{\boldsymbol{a}}$. Hence $\boldsymbol{u} \subseteq \boldsymbol{a}$. Conversely, suppose that $\boldsymbol{u} \subseteq \boldsymbol{a}$. Since $\boldsymbol{u}$ is compact, it can be written in the form $\boldsymbol{u} = \bigvee_{i < n} \Theta_A(x_i, y_i)$, where $n < \omega$ and $x_i, y_i \in A$ for all $i < n$. For all $i < n$, the relation $x_i \equiv y_i \pmod{\boldsymbol{a}}$ holds, so $\varphi \Theta_A(x_i, y_i) = 0$. Joining over $i < n$ yields $\varphi(\boldsymbol{u}) = 0$. $\square$ Claim.

Now let $\boldsymbol{x}, \boldsymbol{y} \in \mathrm{Con}_c\, A$. Since φ is ideal-induced, the following equivalences hold:

$$(\mathrm{Con}_c\, p)(\boldsymbol{x}) = (\mathrm{Con}_c\, p)(\boldsymbol{y}) \Leftrightarrow \boldsymbol{x} \vee \boldsymbol{a} = \boldsymbol{y} \vee \boldsymbol{a} \qquad \text{(see [8, Theorem I.11.3])}$$
$$\Leftrightarrow \varphi(\boldsymbol{x}) = \varphi(\boldsymbol{y}) \qquad \text{(by the Claim above).}$$

In particular, there exists a unique isomorphism $\varepsilon \colon \mathrm{Con}_c\, \overline{A} \to S$ such that $\varphi = \varepsilon \circ \mathrm{Con}_c\, p$.

Now let $f \colon A \to B$ a homomorphism in $\mathcal{V}$ and let $\eta \colon \mathrm{Con}_c\, \overline{A} \to \mathrm{Con}_c\, B$ such that $\mathrm{Con}_c\, f = \eta \circ \mathrm{Con}_c\, p$. For all $x, y \in A$, if $x \equiv y \pmod{\boldsymbol{a}}$, then $p(x) = p(y)$, thus $\Theta_B(f(x), f(y)) = (\mathrm{Con}_c\, f)\Theta_A(x, y) = \eta \Theta_A(p(x), p(y)) = 0$, so $f(x) = f(y)$. Hence $\ker p \subseteq \ker f$, and thus there exists a unique homomorphism $g \colon \overline{A} \to B$ such that $f = g \circ p$. Thus $\eta \circ (\mathrm{Con}_c\, p) = \mathrm{Con}_c\, f = (\mathrm{Con}_c\, g) \circ (\mathrm{Con}_c\, p)$, and hence, since

$\mathrm{Con}_{\mathrm{c}}\, p$ is surjective, $\eta = \mathrm{Con}_{\mathrm{c}}\, g$. Therefore, $\langle p, \varepsilon \rangle$ is a projectability witness for $\langle \varphi, A, S \rangle$.

For $\langle \vee, 0 \rangle$-semilattices S_0 and S_1, the canonical projection from $S_0 \times S_1$ onto S_0 is obviously ideal-induced. Hence, by the result above, the functor $\mathrm{Con}_{\mathrm{c}}$ is projectable. $\square$

We shall now present some applications of Lemma 7.1. These applications will depend on a few results from [31]. We first introduce the following subcategories of $\mathcal{S}$:

- $\mathcal{M}$, the subcategory of all $\langle \vee, 0 \rangle$-semilattices and $\langle \vee, 0 \rangle$-embeddings.
- $\mathcal{S}_{\mathrm{fin}}$, the full subcategory of all finite members of $\mathcal{S}$.
- $\mathcal{S}_{\mathrm{fin}}^*$, the full subcategory of all finite, simple, atomistic lattices and $\langle \vee, 0 \rangle$-homomorphisms.
- $\mathcal{D}$, the full subcategory of all distributive $\langle \vee, 0 \rangle$-semilattices.
- $\mathcal{B}$, the full subcategory of all Boolean $\langle \vee, 0 \rangle$-semilattices.

Now we quote the corresponding results obtained in [31]. While Proposition 7.2 is obtained, in [31], by a simple application of a construction by G. Grätzer and E. T. Schmidt in [11], we know no proof of Proposition 7.3 substantially simpler than the relatively involved categorical argument presented in [31]. The importance of finite simple atomistic lattices (or, more generally, finite simple lattices whose atoms join to the unit) for congruence representation problems of algebras is illustrated by P. P. Pálfy and P. Pudlák [15].

Proposition 7.2. *There exists a functorial retraction $\langle \Upsilon, \xi, \rho \rangle$ of the category $\mathcal{S}_{\mathrm{fin}} \cap \mathcal{M}$ to the full subcategory $\mathcal{S}_{\mathrm{fin}}^* \cap \mathcal{M}$. Furthermore, ξ_K is a $\langle \vee, \wedge, 0, 1 \rangle$-embedding, for every finite $\langle \vee, 0 \rangle$-semilattice K.*

Proposition 7.3. *There exists a functorial retraction $\langle \Phi, \varepsilon, \mu \rangle$ of the category $\mathcal{D}_{\mathrm{fin}} \cap \mathcal{M}$ to the category $\mathcal{B}_{\mathrm{fin}} \cap \mathcal{M}$. Furthermore, ε_A is a $\langle \vee, 0, 1 \rangle$-embedding, for all $A \in \mathrm{Ob}\,\mathcal{D}$.*

Now the promised applications.

Theorem 7.4. *Let $\mathcal{V}$ be a variety of algebras. Then the following statements hold:*
  (i) *If every diagram of finite products of finite atomistic simple lattices and $\langle \vee, 0, 1 \rangle$-embeddings, indexed by a lattice, can be lifted with respect to the $\mathrm{Con}_{\mathrm{c}}$ functor on $\mathcal{V}$, then so can every diagram of finite $\langle \vee, 0, 1 \rangle$-semilattices and $\langle \vee, 0, 1 \rangle$-embeddings, indexed by a lattice.*
  (ii) *If every diagram of finite Boolean $\langle \vee, 0 \rangle$-semilattices and $\langle \vee, 0 \rangle$-embeddings, indexed by a lattice, can be lifted with respect to the $\mathrm{Con}_{\mathrm{c}}$ functor on $\mathcal{V}$, then so can every diagram of finite distributive $\langle \vee, 0 \rangle$-semilattices and $\langle \vee, 0 \rangle$-embeddings, indexed by a lattice.*
  (iii) *The analogue of* (ii) *above for $\langle \vee, 0, 1 \rangle$-embeddings also holds.*

We shall discuss later why we do not formulate the analogue of Theorem 7.4(i) for $\langle \vee, 0 \rangle$-embeddings (although it holds!), see Proposition 7.6.

Proof. We provide a proof for (i); it uses Proposition 7.2. The proofs of (ii) and (iii) are similar. Of course, the proof of (ii) uses Proposition 7.3 and the triple $\langle \Phi, \varepsilon, \mu \rangle$ instead of Proposition 7.2 and the triple $\langle \Upsilon, \xi, \rho \rangle$.

Let $F \colon \mathcal{V} \to \mathcal{S}$ be the functor that with an algebra A associates $\mathrm{Con}_{\mathrm{c}}\, A$ and that with a homomorphism f associates $\mathrm{Con}_{\mathrm{c}}\, f$. Let $\mathcal{I}$ be a lattice, viewed as a category,

and let $\mathbf{D}\colon \mathfrak{I} \to \mathcal{S}_{\mathrm{fin}} \cap \mathcal{M}$ be a $\mathfrak{I}$-indexed diagram of finite $\langle \vee, 0, 1 \rangle$-semilattices and $\langle \vee, 0, 1 \rangle$-embeddings. We define a functor $\mathbb{D}\colon \mathfrak{I} \to \mathrm{Retr}(\mathcal{S}, \mathcal{S})$ by putting

$$\mathbb{D}(x) = \left\langle \mathbf{D}(x), \Upsilon\mathbf{D}(x), \xi_{\mathbf{D}(x)}, \rho_{\mathbf{D}(x)} \right\rangle,$$
$$\mathbb{D}(x \to y) = \langle \mathbf{D}(x \to y), \Upsilon\mathbf{D}(x \to y) \rangle$$

for all $x \leq y$ in $\mathfrak{I}$. Observe that $\mathbf{D}(x) \in \mathrm{Ob}\,\mathcal{S}_{\mathrm{fin}}$, $\Upsilon\mathbf{D}(x) \in \mathrm{Ob}\,\mathcal{S}_{\mathrm{fin}}^*$, and both $\mathbf{D}(x \to y)$ and $\Upsilon\mathbf{D}(x \to y)$ are $\langle \vee, 0, 1 \rangle$-embeddings.

Now we take a closer look at the unfolding $\widetilde{\mathbb{D}}\colon \mathfrak{I} \times \omega \to \mathcal{S}$ of $\mathbb{D}$. Its objects are finite products of finite simple lattices. Its arrows are finite compositions of arrows of the form either σ_n^x (for $x \in \mathfrak{I}$ and $n \in \omega \setminus \{0\}$), which is a *section* of $\mathcal{S}$ (for $\pi_{n+1}^x \circ \sigma_n^x$ is an identity), or $\widehat{\mathbf{D}}^n(x \to y)$, for $x \leq y$ in $\mathfrak{I}$ and $n \in \omega \setminus \{0\}$. As $\widehat{\mathbf{D}}(x \to y) = \Upsilon\mathbf{D}(x \to y)$ is a $\langle \vee, 0, 1 \rangle$-embedding and $f \Pi g$ is a $\langle \vee, 0, 1 \rangle$-embedding whenever both f and g are $\langle \vee, 0, 1 \rangle$-embeddings, $\widehat{\mathbf{D}}^n(x \to y)$ is a $\langle \vee, 0, 1 \rangle$-embedding for all n. Consequently, all arrows of $\widetilde{\mathbb{D}}$ are $\langle \vee, 0, 1 \rangle$-embeddings.

Hence, by assumption, the diagram $\widetilde{\mathbb{D}}$ has a lifting with respect to the $\mathrm{Con}_{\mathrm{c}}$ functor. By Lemma 7.1, the $\mathrm{Con}_{\mathrm{c}}$ functor is projectable. Moreover, $\mathcal{V}$ has arbitrary directed colimits, $\mathrm{Con}_{\mathrm{c}}$ preserves all directed colimits, and $\mathcal{S}$ has finite products. Therefore, by Theorem 6.3, the diagram $\mathbf{D} = \Pi\mathbb{D}$ has a lifting with respect to $\mathrm{Con}_{\mathrm{c}}$. $\qquad\square$

The statement of Theorem 7.4 can be modified in many ways, especially on the category $\mathfrak{I}$ indexing the diagram $\mathbf{D}$. Indeed, the unfolding $\widetilde{\mathbb{D}}$ is indexed by $\mathfrak{I} \times \omega$. So, instead of requiring that $\mathfrak{I}$ be a lattice, we could have required that $\mathfrak{I}$ be a poset, or a distributive lattice, or even an arbitrary category. The only restriction on the arrows of $\mathbf{D}$ is that they are required to be $\langle \vee, 0 \rangle$-*embeddings* ($\langle \vee, 0, 1 \rangle$-embeddings in (i)), in order to be able to apply Propositions 7.2 and 7.3.

On the other hand, although the corresponding analogues of Theorem 7.4 are still valid, they may be so vacuously, that is, their assumptions may not hold.

An example of the latter situation is the analogue of Theorem 7.4(i) for $\langle \vee, 0 \rangle$-embeddings (not necessarily preserving the unit). Let us be more specific.

Example 7.5. Let $\mathcal{V}$ be the variety of all algebras with given similarity type Σ. It follows from results in R. Freese, W. A. Lampe, and W. Taylor [4] that there exists a $\langle \vee, 0 \rangle$-semilattice S that is not isomorphic to $\mathrm{Con}_{\mathrm{c}} A$, for any member A of $\mathcal{V}$. In fact, S may be defined as the $\langle \vee, 0 \rangle$-semilattice of subspaces of an infinite-dimensional vector space over a field with large enough cardinality. On the other hand, S is the directed union of its finite $\langle \vee, 0 \rangle$-subsemilattices. Since the $\mathrm{Con}_{\mathrm{c}}$ functor preserves arbitrary directed colimits, the corresponding diagram of $\langle \vee, 0 \rangle$-semilattices has no lifting with respect to $\mathrm{Con}_{\mathrm{c}}$. This diagram may be taken indexed by the poset of all finite subsets of S, which is a distributive lattice. Therefore, by applying the analogue of Theorem 7.4(i) for $\langle \vee, 0 \rangle$-embeddings, we obtain the following result.

Proposition 7.6. *Let Σ be a similarity type of algebras. Then there exists a diagram, indexed by a distributive lattice, of finite products of finite simple lattices and $\langle \vee, 0 \rangle$-embeddings, without a lifting, with respect to the $\mathrm{Con}_{\mathrm{c}}$ functor, by algebras with similarity type Σ.*

Note that there is a very simple diagram (indexed by a finite category) of finite Boolean $\langle \vee, 0, 1 \rangle$-semilattices and $\langle \vee, 0, 1 \rangle$-embeddings that cannot be lifted, with

respect to the $\mathrm{Con_c}$ functor, by algebras in any given similarity type (see J. Tůma and F. Wehrung, [21, Theorem 8.1] or [23, Introduction]). Furthermore, it is proved in J. Tůma and F. Wehrung [23] that there exists a diagram, indexed by a finite poset, of finite Boolean $\langle \vee, 0, 1 \rangle$-semilattices with $\langle \vee, 0, 1 \rangle$-embeddings, that cannot be lifted, with respect to the $\mathrm{Con_c}$ functor, by lattices and lattice homomorphisms. This is important in telling us how far we may go in the formulation of the coming Proposition 7.8.

Example 7.7. It is still an open problem, dating back to 1945 and usually referred to as CLP, whether every distributive $\langle \vee, 0 \rangle$-semilattice is isomorphic to $\mathrm{Con_c}\, L$ for some lattice L. A survey of some of the latest attempts on that problem can be found in [22]. All these attacks aim at lifting not only *objects* (distributive $\langle \vee, 0 \rangle$-semilattices), but *diagrams* (of finite distributive $\langle \vee, 0 \rangle$-semilattices). Some partial results and many calculations suggest that it is less difficult to lift diagrams of finite Boolean $\langle \vee, 0 \rangle$-semilattices and $\langle \vee, 0 \rangle$-embeddings than to lift diagrams of arbitrary distributive $\langle \vee, 0 \rangle$-semilattices (this is also illustrated in J. Tůma [20]). As every distributive $\langle \vee, 0 \rangle$-semilattice is the directed union of its finite $\langle \vee, 0 \rangle$-subsemilattices (see Fact 4, page 100 in P. Pudlák [17]), a direct application of Theorem 7.4(ii) yields the following.

Proposition 7.8. *If every diagram, indexed by a distributive lattice, of finite Boolean $\langle \vee, 0 \rangle$-semilattices (resp., $\langle \vee, 0, 1 \rangle$-semilattices) with $\langle \vee, 0 \rangle$-embeddings (resp., $\langle \vee, 0, 1 \rangle$-embeddings) can be lifted, with respect to the $\mathrm{Con_c}$ functor, by lattices and lattice homomorphisms, then every distributive $\langle \vee, 0 \rangle$-semilattice (resp., $\langle \vee, 0, 1 \rangle$-semilattice) is isomorphic to $\mathrm{Con_c}\, L$ for some lattice L.*

Various other formulations are possible, for example with $\langle \vee, 0 \rangle$-semilattices and 0-lattice homomorphisms, or bounded $\langle \vee, 0, 1 \rangle$-semilattices and bounded lattices, with similar proofs. However, it is an open problem whether any of the corresponding assumptions actually holds.

8. Nonstable K-theory of von Neumann regular rings

For a (unital, associative) ring R, let $\mathrm{FP}(R)$ denote the class of all finitely generated projective right R-modules. For $X \in \mathrm{FP}(R)$, denote by $[X]$ the isomorphism class of X. Isomorphisms classes can be added, with the rule $[X] + [Y] = [X \oplus Y]$, and the monoid $V(R)$ of all isomorphism classes of members of $\mathrm{FP}(R)$ is a conical commutative monoid with order-unit.

Now we specialize our class of rings. A ring R is (von Neumann) *regular*, if it satisfies the axiom $(\forall x)(\exists y)(xyx = x)$. For a regular ring R, the monoid $V(R)$ is a *refinement monoid*, see K. R. Goodearl [5, Theorem 2.8]. However, there exists a conical refinement monoid (and even the positive cone of a dimension group) that is not isomorphic to $V(R)$, for any regular ring R, see F. Wehrung [26]. This counterexample has cardinality $\aleph_2$. The corresponding problem for smaller cardinalities is still open, and probably very difficult, even in the countable case, see the fundamental open problem in K. R. Goodearl [6]. We formulate the corresponding question for the countable case: *Is every countable conical refinement monoid with order-unit isomorphic to $V(R)$ for some regular ring R?*

In the present section we wish to provide a new insight on this problem, based on Theorem 6.3. First of all, observe that for regular rings R and S, every ring homomorphism $f \colon R \to S$ induces a unique monoid homomorphism $V(f) \colon V(R) \to$

$V(S)$ such that $V(f)([xR]) = [f(x)S]$, for all $x \in R$. Hence V is a *functor* from the category $\mathcal{R}$ of regular rings and (unital) ring homomorphisms to the category $\mathcal{M}$ of conical refinement monoids with order-unit and unit-preserving homomorphisms. It is well-known that $\mathcal{R}$ has arbitrary directed colimits and that the V functor preserves all directed colimits.

We need the following lemma.

Lemma 8.1. *Let R be a regular ring, let M be a conical refinement monoid, and let $\varphi\colon V(R) \twoheadrightarrow M$ be an ideal-induced monoid homomorphism. Then the triple $\langle \varphi, R, M \rangle$ has a projectability witness with respect to the functor V. In particular, the functor V is projectable.*

Proof. Put $J = \varphi^{-1}\{0\}$ and $I = \{x \in R \mid [xR] \in J\}$. So I is a two-sided ideal of R and $J = \{[xR] \mid x \in I\}$, see Corollary 4.4 and Proposition 4.6 in F. Wehrung [27]. Put $\overline{R} = R/I$ and denote by $p\colon R \twoheadrightarrow R/I$ the canonical projection. Since p is surjective, it is an epic. Denote by $\pi\colon V(R) \twoheadrightarrow V(R)/J$ the canonical projection. By describing explicitly the isomorphism $V(R/I) \cong V(R)/V(I)$ (see Proposició 4.1.6 and Teorema 4.1.7 in E. Pardo [16], or P. Ara *et al.* [1, Proposition 1.4]), we obtain that there exists an isomorphism $\zeta\colon V(\overline{R}) \to V(R)/J$ such that

$$\zeta\big([(x+I)\overline{R}]\big) = \pi\big([xR]\big), \text{ for all } x \in R.$$

Hence $\zeta \circ V(p) = \pi$. Since φ is ideal-induced, it induces a monoid isomorphism $\psi\colon V(R)/J \to M$ such that $\psi \circ \pi = \varphi$. Hence $\varepsilon = \psi \circ \zeta$ is an isomorphism from $V(\overline{R})$ onto M such that $\varphi = \varepsilon \circ V(p)$.

Finally, let S be a regular ring, let $f\colon R \to S$ be a ring homomorphism, and let $\eta\colon V(\overline{R}) \to V(S)$ be a monoid homomorphism such that $V(f) = \eta \circ V(p)$. From the obvious fact that $x = 0$ iff $[xR] = 0$ (in any ring), we obtain that $\ker p$ is contained in $\ker f$, thus there exists a unique ring homomorphism $g\colon \overline{R} \to S$ such that $g \circ p = f$. Hence, $V(g) \circ V(p) = V(f) = \eta \circ V(p)$, and hence, since $V(p)$ is surjective, $V(g) = \eta$. Therefore, $\langle p, \varepsilon \rangle$ is a projectability witness for $\langle \varphi, R, M \rangle$.

For commutative monoids M_0 and M_1, the canonical projection from $M_0 \times M_1$ onto M_0 is obviously ideal-induced. Hence, by the result above, the functor V is projectable. $\square$

Applying Theorem 6.3, and then Lemma 8.1 just as Lemma 7.1 is used in the proof of Theorem 7.4, yields the following result.

Proposition 8.2. *Let M be a conical refinement monoid with order-unit and let $\mathcal{M}'$ be a full subcategory of $\mathcal{M}$ satisfying the following conditions:*

(i) *$\mathcal{M}'$ is closed under finite nonempty products.*
(ii) *Every diagram, indexed by $\omega \times \omega$, of members of $\mathcal{M}'$ with embeddings (see Section 2), has a lifting, with respect to the V functor, by regular rings.*
(iii) *There exists a ω-indexed diagram in $\mathrm{Retr}(\mathcal{M}, \mathcal{M}')$ of the form*

$$
\begin{array}{ccccccc}
N_0 & \overset{g_0}{\hookrightarrow} & N_1 & \overset{g_1}{\hookrightarrow} & N_2 & \overset{g_2}{\hookrightarrow} & \cdots \quad \cdots \\
\Updownarrow & & \Updownarrow & & \Updownarrow & & \\
M_0 & \underset{f_0}{\hookrightarrow} & M_1 & \underset{f_1}{\hookrightarrow} & M_2 & \underset{f_2}{\hookrightarrow} & \cdots \quad \cdots ,
\end{array}
$$

with all f_n-s and g_n-s embeddings, such that $\varinjlim_n M_n \cong M$.

Then M is isomorphic to $V(R)$, for some regular ring R.

However, unlike the situation with finite distributive and Boolean semilattices (see Proposition 7.8), we do not have at the present time any reasonable candidate for the category $\mathcal{M}'$. A possibility would be to use colimits of diagrams of the form

$$F \overset{f}{\hookrightarrow} F \overset{f}{\hookrightarrow} F \overset{f}{\hookrightarrow} \cdots \cdots,$$

where F is a finitely generated commutative monoid and $f \colon M \hookrightarrow M$ is an embedding from M into M (we need to ensure that the colimit has refinement). A related construction is used, in C. Moreira dos Santos [14], to find a refinement monoid whose maximal antisymmetric quotient is not a refinement monoid. Also, our insistence on *embeddings* instead of just one-to-one monoid homomorphisms is purely experimental—in particular, the analogue of Proposition 8.2 for one-to-one monoid homomorphisms is also valid.

9. Concluding remarks

Sections 7 and 8 were written with some amount of detail, in particular proving the projectability of both functors $\mathrm{Con}_{\mathrm{c}}$ (on algebras) and V (on regular rings). There are many other situations in which the problem of the range of a given projectable functor is raised. Let us mention a few more examples of projectable functors and some of the corresponding problems:

— The functor $\boldsymbol{L}$ from von Neumann regular rings to complemented modular lattices. For each regular ring R, $\boldsymbol{L}(R)$ is the lattice of all principal right ideals of R. Determining the range of the $\boldsymbol{L}$ functor is a hard open problem, initiated by von Neumann's Coordinatization Theorem. Recently, the author proved that even for countable regular rings, the range of $\boldsymbol{L}$ is not a first-order class, see F. Wehrung [30].

— The forgetful functor that with every modular ortholattice L associates the corresponding lattice. Some partial results about the range of that functor are established in [30].

— The functor Dim that with a lattice L associates its *dimension monoid* $\mathrm{Dim}\,L$, see F. Wehrung [28]. Determining the range of Dim seems to be an even harder problem than CLP (see Section 7). When restricted to complemented modular lattices, it is strongly related to the $V(R)$ representation problem discussed in Section 8.

— The functor ∇ on refinement monoids. For a refinement monoid M, $\nabla(M)$ is the maximal semilattice quotient of M; it is a distributive $\langle \vee, 0\rangle$-semilattice. The problem of the determination of the range of ∇ on certain classes of refinement monoids was initiated in K. R. Goodearl and F. Wehrung [7]. An important milestone is P. Růžička's result that provides a distributive $\langle \vee, 0, 1\rangle$-semilattice of cardinality $\aleph_2$ which is not isomorphic to $\nabla(G^+)$ for any dimension group G, see P. Růžička [18]. This result was later improved by the author to the cardinality $\aleph_1$ and larger classes of monoids, see F. Wehrung [29].

W. A. Lampe proved in [12] a result that implies that every $\langle \vee, 0, 1\rangle$-semilattice is isomorphic to $\mathrm{Con}_{\mathrm{c}}\,G$ for some groupoid (i.e., nonempty set with a binary operation) G. The proof presented in [12] is apparently not functorial. This raises the question whether the assumption of Theorem 7.4(i) holds for some variety. The boldest that we may ask in that direction is the following.

Problem 1. Denote by $\mathcal{G}$ the category of all groupoids and groupoid homomorphisms. Does there exist a functor Ψ from the category of $\langle \vee, 0, 1 \rangle$-semilattices with $\langle \vee, 0, 1 \rangle$-embeddings to the category $\mathcal{G}$ such that $\mathrm{Con_c}\,\Psi$ is equivalent to the identity?

By a suitable analogue of Theorem 7.4(i), it is sufficient to solve the problem on the category of finite products of simple lattices with $\langle \vee, 0, 1 \rangle$-embeddings.

A related problem is the following.

Problem 2. Can every finite diagram of finite $\langle \vee, 0 \rangle$-semilattices with $\langle \vee, 0 \rangle$-embeddings be lifted, with respect to the $\mathrm{Con_c}$ functor, by a diagram of algebras (in some similarity type)?

Our next problem is related to lifting problems of distributive *lattices* with zero by various functors. Some known results are the following:

- Every distributive lattice D with zero is isomorphic to $\nabla(\mathbb{Q}\langle D \rangle^+)$, for a certain dimension vector space $\mathbb{Q}\langle D \rangle$ ("temperate power") that is easily seen to depend functorially on D; see [7].
- Every distributive lattice D with zero is isomorphic to $\mathrm{Con_c}\,L_D$, for a lattice L_D that is a directed $\langle \vee, \wedge, 0 \rangle$-colimit of finite atomistic lattices and that depends functorially on D; see [17].
- Every distributive lattice D with zero is isomorphic to the compact (i.e., finitely generated) ideal lattice $\mathrm{Id_c}\,R$ of some locally matricial ring R (over any given field); see P. Růžička [19].

A ring is *locally matricial*, if it is a directed colimit of a diagram of finite products of the form $\prod_{i<n} \mathrm{M}_{m_i}(F)$, where F is a given field, n and the m_i-s are positive integers, and $\mathrm{M}_{m_i}(F)$ denotes the $m_i \times m_i$ full matrix ring over F. We ask whether "the best of all worlds" is reachable:

Problem 3. Let F be a field. Does there exist a functor Φ, from distributive 0,1-lattices with $\langle \vee, \wedge, 0, 1 \rangle$-embeddings to locally matricial rings over F with (unital) ring homomorphisms, such that $\mathrm{Id_c}\,\Phi$ is equivalent to the identity?

In view of some results of [7], it is reasonable to ask whether if such a functor existed, the composition $K_0\Phi$ could be equivalent to the functor $\mathbb{Q}\langle\_\rangle$.

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